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Weak compressibility of surface wave turbulence

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We study the growth of small-scale inhomogeneities in the density of particles floating in weakly nonlinear small-amplitude surface waves. Despite the small amplitude, the accumulated effect of the long-time evolution may produce a strongly inhomogeneous distribution of the floaters: density fluctuations grow exponentially with a small but finite exponent. We show that the exponent is of sixth or higher order in wave amplitude. As a result, the inhomogeneities do not form within typical time scales of the natural environment. We conclude that the turbulence of surface waves is weakly compressible and alone it cannot be a realistic mechanism of the clustering of matter on liquid surfaces.

1. Introduction

Clustering of matter on the surface of lakes and pools and of oil slicks and seaweed on the sea surface is well-known empirically but there is no theory that describes it. Since surface flows are compressible even for incompressible fluids, such a theory should be based on the general description of the development of density inhomogeneities in a compressible flow. An important characteristic of the formation of small-scale inhomogeneities is the negative of the sum of the Lyapunov exponents of the flow, λ. It gives the asymptotic logarithmic growth rate of the density on fluid particle trajectories at large times. The rate λ is the negative of the average value of the velocity divergence seen by a fluid particle, and it is always non-negative because contracting regions with negative divergence have more particles and hence larger statistical weight, see Balkovsky, Falkovich & Fouxon (2001) and Ruelle (1996, 1997, 1999). We note that λ is also the production rate of the Gibbs entropy, so that the condition $\lambda \geqslant 0$ can be regarded as an analogue of the second law of thermodynamics for the dissipative dynamics, see Ruelle (1996, 1997, 1999) and Falkovich & Fouxon (2003, 2004). For a generic flow $\lambda > 0$ and the asymptotic density becomes a singular measure, the so-called Sinai–Ruelle–Bowen measure, see e.g. Dorfman (1999). For floaters this means that they form a multi-fractal structure on the surface, see Yu, Ott & Chen (1991), Falkovich, Gawędzki & Vergassola (2001), Balkovsky *et al.* (2001), Ruelle (1999), Falkovich & Fouxon (2003, 2004), Bec, Gawędzki & Horvai (2004), Balk, Falkovich & Stepanov (2004) and Eckhardt & Schumacher (2001) for theory and Ramshankar, Berlin & Gollub (1990), Sommerer & Ott (1993), Sommerer (1996), Nameson, Antonsen & Ott (1996), Schroder *et al.* (1996), Cressman & Goldburg (2003), Denissenko, Falkovich & Lukaschuk (2006), Bandi, Goldburg & Cressman (2006) for experiments. This structure is the attractor of the two-dimensional dissipative dynamics obeyed by the particles on the surface, and it evolves constantly for time-dependent flows, see e.g. Dorfman (1999) and

Ott (2002). The Kaplan–Yorke dimension of the attractor, $D_{KY} = 1 + \lambda_1/|\lambda + \lambda_1|$, is between one and two, assuming that the principal Lyapunov exponent λ_1 is positive, see Ott (2002).

Surface flows are generic compressible flows for which the Eulerian compressibility, measured by the dimensionless ratio $C = \langle(\partial_i v_i)^2\rangle/\langle(\partial_j v_i)^2\rangle$, is of order one, cf. Boffetta *et al.* (2006*a*). Here angular brackets stand for the spatial average, $\boldsymbol{v}$ is the floater velocity field, and C changes from zero for incompressible flow to one for potential flow. For $C \sim 1$ it is expected that $\lambda \sim \lambda_1$, so that the deviation of D_{KY} from the surface dimension 2 is also of order one, $2 - D_{KY} \sim 1$. The expectation holds for the flow on the surface of three-dimensional turbulence. Performing numerical simulations with the full three-dimensional Navier–Stokes equations, Boffetta *et al.* (2006*a*) found $C \approx 0.5$, $D_{KY} \approx 1.15$ and observed strong clustering on the surface, see also Cressman *et al.* (2004), Boffetta, Davoudi & Lillo (2006*b*) and Eckhardt & Schumacher (2001). However, underwater turbulence is relatively rare in the natural environment (due to stable stratification), and it is important to consider other surface flows, of which small-amplitude surface waves are probably the most widespread. Despite the small, amplitude a small-but-finite λ produces a large effect over time scales of order $1/\lambda$ and larger. Thus to estimate the role of surface waves in the formation of the floater inhomogeneities in the natural environment, one needs to know how small λ is. In this article we show that λ is of sixth or higher order in wave amplitude. Note that for surface waves the degree of compressibility C is due to linear waves, which produce potential flow with $C = 1$. Thus one could expect that the estimates $\lambda \sim \lambda_1$ and $2 - D_{KY} \sim 1$ would hold for surface waves as they do for the underwater turbulence. We show that, under some natural non-degeneracy assumptions described in § 5, for surface waves $\lambda \ll \lambda_1$.

The calculation of λ for random waves in different situations was considered in Balk *et al.* (2004) and Vucelja, Falkovich & Fouxon (2007). For surface waves, at the linear order in the wave amplitude, the particles move periodically, see e.g. Batchelor (1967), hence there is no net clustering. Thus analysis of clustering of the floaters requires including the nonlinear effects. Balk *et al.* (2004) assumed a linear relation between the velocity field of the floaters and the wave amplitudes, and considered a Gaussian ensemble of non-interacting waves. The nonlinearity in this case comes from expressing the Lagrangian objects in terms of the Eulerian ones. It was shown that λ vanishes at the fourth order in the wave amplitude for longitudinal waves, whose dispersion relation does not allow the same frequency for two different wave vectors (e.g. sound, gravity, capillary waves). On the other hand, the lowest-order non-vanishing contribution to λ_1 was shown to be always of the fourth order in the wave amplitude. Under the same assumption of a linear relation between the velocity and the wave amplitudes, Vucelja *et al.* (2007) demonstrated that taking account of the wave interactions does not change the conclusion of Balk *et al.* (2004) on the vanishing of λ at the fourth order in the wave amplitude.

The results above are inconclusive as far as surface waves are concerned, for which the relation between the velocity and the amplitudes is nonlinear due to a small but finite curvature of the surface, see Zakharov (1966, 1968). For this case a separate calculation of λ is needed. Here we provide such a calculation. We consider weakly nonlinear surface waves and show that neither the wave interactions, nor the nonlinear relation between the velocity and the amplitudes, create a non-zero sum of the Lyapunov exponents up to the fourth order in the wave amplitudes. The main tool of our analysis is a recently derived Green–Kubo-type formula for the sum of the Lyapunov exponents, see Falkovich & Fouxon (2004, 2003). This formula expresses λ in terms of the correlations of the flow divergence in the particle frame. It describes the interplay between the particle motion and the local flow compressions in accumulating density inhomogeneities which become pronounced as a result of the long-time evolution. The slowness of the particle drift from its initial position allows us to express the correlations in terms of the Eulerian correlation functions of the velocity, which we further evaluate by a lengthy but straightforward calculation, cf. Vucelja *et al.* (2007).

The text is organized as follows. In the next Section we introduce the expression for λ in terms of the Eulerian correlations of the surface flow velocity, valid up to the fourth order in the wave amplitude. To perform the calculation we need to express the velocity in terms of the normal coordinates. This is dealt with in § 3. The calculation of the various terms occurring in λ is performed in § 4. A discussion finishes the article.

2. The sum of the Lyapunov exponents

The behaviour of the density $n(t, \boldsymbol{x})$ in a velocity field $\boldsymbol{v}(t, \boldsymbol{x})$ is governed by the continuity equation $\partial_t n + \nabla\cdot(\boldsymbol{v}n) = 0$, see Batchelor (1967). Introducing Lagrangian particle trajectories by the equation $\partial_t \boldsymbol{X}(t, \boldsymbol{x}) = \boldsymbol{v}[t, \boldsymbol{X}(t, \boldsymbol{x})]$ with the initial condition $\boldsymbol{X}(0, \boldsymbol{x}) = \boldsymbol{x}$, we may write the solution for the density as

$$n[t, \boldsymbol{X}(t, \boldsymbol{x})] = n(0, \boldsymbol{x}) \exp\left[-\int_0^t J[t', \boldsymbol{X}(t', \boldsymbol{x})]\mathrm{d}t'\right].$$

Here we have introduced $J(t, \boldsymbol{x}) \equiv \nabla\cdot\boldsymbol{v}(t, \boldsymbol{x})$. We characterize the growth of spatial inhomogeneities by the asymptotic logarithmic growth rate λ at large times, defined by

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln\left[\frac{n[t, \boldsymbol{X}(t, \boldsymbol{x})]}{n(0, \boldsymbol{x})}\right] = -\lim_{t\to\infty} \frac{1}{t} \int_0^t J[t', \boldsymbol{X}(t', \boldsymbol{x})]\mathrm{d}t'. \tag{2.1}$$

The above limit is well-defined, Dorfman (1999), and it gives the negative of the sum of the Lyapunov exponents of the flow $\boldsymbol{v}(t, \boldsymbol{x})$. It was shown in Falkovich & Fouxon (2004, 2003) that if $\boldsymbol{v}(t, \boldsymbol{x})$ is a random, spatially homogeneous, stationary flow, then

$$\lambda = \int_0^\infty \mathrm{d}t \langle J(0, \boldsymbol{x}) J[t, \boldsymbol{X}(t, \boldsymbol{x})]\rangle. \tag{2.2}$$

We shall apply the above formula to the case where $\boldsymbol{v}(t, \boldsymbol{x})$ is the two-dimensional velocity field governing the motion of the floaters in a (quasi-)stationary ensemble of weakly nonlinear surface waves sustained by some forcing, see Zakharov, L'vov & Falkovich (1992). We first use the smallness of the amplitude to express the Lagrangian correlation function in (2.2) in terms of the velocity correlation functions given in the Eulerian frame. We follow Vucelja *et al.* (2007) who considered (2.2) in the case of arbitrary low-amplitude waves. For the dispersion relation Ω_k, considering packets with both the wavenumber and the width of order k, the correlation time of w can be estimated as Ω_k^{-1} and the correlation length as k^{-1}. The particle deviation from the initial position, $\boldsymbol{X}(t, \boldsymbol{x}) - \boldsymbol{x}$, during the period $t \simeq \Omega_k^{-1}$, is $\epsilon = kv/\Omega_k \ll 1$ times smaller than k^{-1} which allows expansion of (2.2) near $\boldsymbol{x}$. Performing the expansion to order ϵ^4 we find

$$\lambda \approx \lambda_2 + \lambda_3 + \lambda_4, \tag{2.3}$$

$$\lambda_2 \equiv \frac{1}{2}\int \mathrm{d}t \langle J(0)J(t)\rangle, \quad \lambda_3 \equiv \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}t_1 \left\langle J(0)\frac{\partial J(t)}{\partial x^\alpha} v^\alpha(t_1)\right\rangle, \tag{2.4}$$

$$\lambda_4 \equiv \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \left\langle J(0)v_\beta(t_2)\left(\frac{\partial J(t)}{\partial x_\alpha}\frac{\partial v_\alpha(t_1)}{\partial x_\beta} + \frac{\partial^2 J(t)}{\partial x_\alpha \partial x_\beta}v_\alpha(t_1)\right)\right\rangle. \quad (2.5)$$

Here and below we suppress the spatial coordinate $\boldsymbol{x}$ over which the averaging is performed. The expansion above was introduced in Vucelja *et al.* (2007). Note that all contributions λ_i are of fourth or higher order in wave amplitude, see below. To use the above formula to find λ to order ϵ^4, we need to establish the expression for the surface flow $\boldsymbol{v}$ to order ϵ^3.

3. The velocity field of the floaters

The velocity field that governs the evolution of the floater coordinates $\boldsymbol{r} = (x, y)$ in the horizontal plane has the following form:

$$\boldsymbol{v}(\boldsymbol{r}, t) = \left(\frac{\partial \phi(\boldsymbol{r}, z, t)}{\partial x}[z = \eta(\boldsymbol{r}, t)], \frac{\partial \phi(\boldsymbol{r}, z, t)}{\partial y}[z = \eta(\boldsymbol{r}, t)]\right), \quad (3.1)$$

where $\eta(\boldsymbol{r}, t)$ is the surface elevation and $\phi(\boldsymbol{r}, z, t)$ is the velocity potential, $\boldsymbol{v} = \nabla\phi$. Zakharov (1966, 1968, 1999) showed that the system of weakly interacting surface waves is a Hamiltonian system with canonically conjugate coordinates $\eta(\boldsymbol{r}, t)$ and $\psi(\boldsymbol{r}, t) \equiv \phi(\boldsymbol{r}, \eta(\boldsymbol{r}, t), t)$. The calculation of the surface flow of the floaters to order ϵ^3, needed for calculation of λ to order ϵ^4, is given in Appendix A. The result is

$$\boldsymbol{v} = \mathrm{i}\int \frac{\mathrm{d}\boldsymbol{k}_1}{(2\pi)^2}\boldsymbol{k}_1 \exp[\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}]\psi_1 - \mathrm{i}\int \frac{\mathrm{d}\boldsymbol{k}_{12}}{(2\pi)^4}\exp[\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{r}]\overline{|k_1|}\boldsymbol{k}_2\psi_1\eta_2$$
$$-\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}\boldsymbol{k}_{123}}{(2\pi)^6}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\cdot\boldsymbol{r}}\psi_1\eta_2\eta_3\left(|k_1|^2\boldsymbol{k}_2 + |k_1|^2\boldsymbol{k}_3 - 2\overline{\sqrt{k_1^2+k_2^2}}\,\overline{|k_1|}\boldsymbol{k}_3\right), \quad (3.2)$$

where h is the fluid depth and we introduce the shorthand notation $\eta_i(t) = \eta(\boldsymbol{k}_i, t)$, $\psi_i(t) = \psi(\boldsymbol{k}_i, t)$, $\mathrm{d}\boldsymbol{k}_{ijl\ldots} = \mathrm{d}\boldsymbol{k}_i\mathrm{d}\boldsymbol{k}_j\mathrm{d}\boldsymbol{k}_l\ldots$ and $\overline{|k|} = |k|\tanh(|k|h)$. In the approximation of infinitely deep fluid, $h \to \infty$, the above formula corresponds to formula (1.8) from Zakharov (1968). Note that the velocity field on the surface $\boldsymbol{v}(\boldsymbol{r}, t)$ is neither potential nor solenoidal.

In the calculation of λ in the following Sections we will use a basic statistical property of the wave turbulence – its approximate Gaussianity, see e.g. Zakharov *et al.* (1992), Longuet-Higgins (1963) and Chelton & Eddy (1993) and references therein. To leading order in the small wave amplitude, the correlation functions of η and ψ can be calculated using Wick's theorem for Gaussian statistics (Wick's theorem is reproduced in Appendix B). Gaussianity is most succinctly expressed in terms of the normal coordinates $a(\boldsymbol{k}, t)$ defined by

$$\eta(\boldsymbol{k}, t) = \sqrt{\frac{|k|}{2\Omega_k}}[a(\boldsymbol{k}, t) + a^*(-\boldsymbol{k}, t)], \quad \psi(\boldsymbol{k}, t) = -\mathrm{i}\sqrt{\frac{\Omega_k}{2|k|}}[a(\boldsymbol{k}, t) - a^*(-\boldsymbol{k}, t)], \quad (3.3)$$

where Ω_k is the dispersion relation: $\Omega_k^2 = |\boldsymbol{k}|(g + (\sigma/\rho)|\boldsymbol{k}|^2)\tanh[|\boldsymbol{k}|h]$, where g is the gravitational acceleration, σ is the surface tension and ρ is the density of the fluid, see Zakharov (1999). Then in the Gaussian approximation the pair correlation functions are given by

$$\langle a^*(\boldsymbol{k}, t)a(\boldsymbol{k}', 0)\rangle = (2\pi)^2\delta(\boldsymbol{k} - \boldsymbol{k}')n(\boldsymbol{k})\exp[\mathrm{i}\Omega_k t], \quad \langle a(\boldsymbol{k}, t)a(\boldsymbol{k}', 0)\rangle = 0, \quad (3.4)$$

$$\left.\begin{aligned}
\langle\psi(\boldsymbol{k},t)\psi(\boldsymbol{k}',0)\rangle &= \frac{\Omega_k(2\pi)^2\delta(\boldsymbol{k}+\boldsymbol{k}')}{2k}[n(\boldsymbol{k})\exp(-\mathrm{i}\Omega_{\boldsymbol{k}}t)+n(-\boldsymbol{k})\exp(i\Omega_{-\boldsymbol{k}}t)],\\
\langle\eta(\boldsymbol{k},t)\eta(\boldsymbol{k}',0)\rangle &= \frac{k(2\pi)^2\delta(\boldsymbol{k}+\boldsymbol{k}')}{2\Omega_{\boldsymbol{k}}}[n(\boldsymbol{k})\exp(-\mathrm{i}\Omega_{\boldsymbol{k}}t)+n(-\boldsymbol{k})\exp(\mathrm{i}\Omega_{-\boldsymbol{k}}t)],\\
\langle\psi(\boldsymbol{k},t)\eta(\boldsymbol{k}',0)\rangle &= \frac{(2\pi)^2\delta(\boldsymbol{k}+\boldsymbol{k}')}{2i}[n(\boldsymbol{k})\exp(-\mathrm{i}\Omega_{\boldsymbol{k}}t)-n(-\boldsymbol{k})\exp(\mathrm{i}\Omega_{-\boldsymbol{k}}t)].
\end{aligned}\right\} \quad (3.5)$$

We now return to the expression for the sum of the Lyapunov exponents (2.3).

4. Calculation of the sum of the Lyapunov exponents

In this Section we provide the calculation of λ to the fourth order in wave amplitude, based on the calculation of the different contributions λ_i, see (2.3)–(2.5). Some parts of this analysis deal with subjects already considered in Balk *et al.* (2004) and Vucelja *et al.* (2007); however our analysis is different and uses specific properties of surface waves. Below we provide a more detailed calculation, than in the short articles Balk *et al.* (2004) and Vucelja *et al.* (2007).

The general structure of the calculation is as follows. Substituting the expression (3.2) for the velocity into λ_i, gives an expression for λ as a sum of the terms involving products of two, three and four fields. The latter terms are already of the fourth order in wave amplitude in the Gaussian approximation. Thus for them one can directly use Wick's theorem to express the answer in terms of the pair correlation functions given by (3.5). As an example of such a computation, below we calculate λ_4 which contains the terms with four fields only. Also the calculation of λ_4 is of particular interest as will become clear in the end of the next subsection.

4.1. *Calculation of* λ_4

We consider the contribution λ_4 to λ. To calculate λ_4 to the fourth order in wave amplitude we may assume Gaussian non-interacting waves and use Wick's theorem to decouple the averages. Employing identities such as $\langle v_\alpha(t_1)\partial_\alpha\partial_\beta J(t)\rangle = -\langle(\partial_\beta v_\alpha(t_1))(\partial_\alpha J(t))\rangle$, that follow by integration by parts, one finds

$$\begin{aligned}
\lambda_4 = &-\int_0^\infty \mathrm{d}t\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\left[\left\langle J(0)\frac{\partial J(t)}{\partial x_\alpha}\right\rangle\langle v_\alpha(t_1)J(t_2)\rangle+\left\langle J(0)\frac{\partial v_\alpha(t_1)}{\partial x_\beta}\right\rangle\right.\\
&\left.\times\left\langle J(t)\frac{\partial v_\beta(t_2)}{\partial x_\alpha}\right\rangle+\left\langle\frac{\partial J(0)}{\partial x_\alpha}\frac{\partial J(t)}{\partial x_\beta}\right\rangle\langle v_\alpha(t_1)v_\beta(t_2)\rangle+\left\langle J(t_2)\frac{\partial J(t)}{\partial x_\alpha}\right\rangle\langle v_\alpha(t_1)J(0)\rangle\right].
\end{aligned} \quad (4.1)$$

Here we do not assume isotropy of the waves. Isotropy would make terms such as $\langle v_\alpha(t_1)J(t_2)\rangle$ vanish. At the considered order, $\boldsymbol{v}=\nabla\psi$ is a potential field and a spectral representation of the pair-correlation function gives

$$\langle\psi(0)\psi(t)\rangle = \int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^2}E(\boldsymbol{k})\cos(\Omega_{\boldsymbol{k}}t) \Rightarrow \langle v_\alpha(0)v_\beta(t)\rangle = \int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^2}k_\alpha k_\beta E(\boldsymbol{k})\cos(\Omega_{\boldsymbol{k}}t),$$

where $E(\boldsymbol{k})$ is expressible in terms of $n(\boldsymbol{k})$ in (3.5). Similar expressions hold for other correlation functions in (4.1). Note that the potentiality of surface waves, holding in the Gaussian approximation, makes the velocity spectrum vanish at $k=0$ even if

$E(k=0)\neq 0$ (see Balk *et al.* 2004; Vucelja *et al.* 2007). We find

$$\lambda_4=\int\frac{\mathrm{d}\boldsymbol{k}\mathrm{d}\boldsymbol{q}\,E(\boldsymbol{k})E(\boldsymbol{q})}{(2\pi)^4}\int_0^\infty \mathrm{d}t\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\Big[k^4q^2(\boldsymbol{k}\cdot\boldsymbol{q})\sin[\Omega_{\boldsymbol{k}}t]\sin[\Omega_{\boldsymbol{q}}(t_1-t_2)]$$
$$-k^2q^2(\boldsymbol{k}\cdot\boldsymbol{q})^2\cos[\Omega_{\boldsymbol{k}}t_1]\cos[\Omega_{\boldsymbol{q}}(t-t_2)]-k^4(\boldsymbol{k}\cdot\boldsymbol{q})^2\cos[\Omega_{\boldsymbol{k}}t]\cos[\Omega_{\boldsymbol{q}}(t_1-t_2)]$$
$$+k^4q^2(\boldsymbol{k}\cdot\boldsymbol{q})\sin[\Omega_{\boldsymbol{k}}(t-t_2)]\sin[\Omega_{\boldsymbol{q}}t_1]\Big]. \tag{4.2}$$

To calculate the time integrals we represent the products of the trigonometric functions above as sums or differences of cosine functions and use

$$\left.\begin{aligned}\int_0^\infty \mathrm{d}t\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cos(at+bt_1+ct_2)&=-\frac{\pi\delta(a)}{b(b+c)}+\frac{\pi\delta(a+b)}{bc}-\frac{\pi\delta(a+b+c)}{c(b+c)},\\ \int_0^\infty \mathrm{d}t\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cos(at-bt_1+bt_2)&=-\frac{\pi\delta'(a)}{b}+\frac{\pi\delta(a)}{b^2}-\frac{\pi\delta(a-b)}{b^2},\ \ b\neq -c.\end{aligned}\right\} \tag{4.3}$$

All terms that are supported only at the zero frequency in the frequency representation (δ-functions or their derivatives) are also supported only at the zero wavenumber, since the dispersion relation of surface waves vanishes at $k=0$ only. As a result, due to the presence of positive powers of k in (4.2), these terms vanish (similarly to the vanishing of the velocity spectrum at $k=0$ shown above). It is then easy to see that the first and the fourth terms in λ_4 (having the same dependence on the wave vectors) cancel each other, while the second and the third terms give

$$\lambda_4=\int\frac{\mathrm{d}\boldsymbol{k}\mathrm{d}\boldsymbol{q}}{(2\pi)^4}E(\boldsymbol{k})E(\boldsymbol{q})k^2(\boldsymbol{k}\cdot\boldsymbol{q})^2(k^2-q^2)\left(\frac{\pi\delta(\Omega_{\boldsymbol{k}}-\Omega_{\boldsymbol{q}})}{2\Omega_{\boldsymbol{q}}^2}\right). \tag{4.4}$$

Since for surface waves the equality of the frequencies of two waves implies the equality of their wavelengths, then the above terms cancel each other, and $\lambda_4=0$. This reproduces in a simple way the result of Balk *et al.* (2004) that in the Gaussian approximation λ vanishes for potential waves having the property that the equality of the frequencies implies the equality of the wavelengths.

4.2. *Reduction to terms involving wave interactions and the importance of the zero frequency field*

Having shown $\lambda_4=0$, let us consider λ_2 and λ_3. The terms in λ_2 and λ_3 that contain products of four fields can be calculated along the same lines as the calculation of λ_4 in the previous subsection. The calculations are given in Appendix C and they show that these terms vanish identically, just like λ_4. We are left with (see (C 6) from Appendix C)

$$\lambda=\frac{1}{2}\int\frac{k_1^2k_2^2\mathrm{d}\boldsymbol{k}_{12}\mathrm{d}\omega}{(2\pi)^5}\langle\psi_1(\omega)\psi_2(\omega=0)\rangle$$
$$-\int\frac{\mathrm{d}\boldsymbol{k}_{123}\mathrm{d}\omega_{12}}{(2\pi)^8}|k_1|k_3^2\big(\boldsymbol{k}_1\cdot\boldsymbol{k}_2+k_2^2\big)\langle\psi_1(\omega_1)\eta_2(\omega_2)\psi_3(\omega=0)\rangle$$
$$-\int\frac{\mathrm{d}\boldsymbol{k}_{123}\mathrm{d}\omega_{123}}{(2\pi)^9}k_1^2k_2^2(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)\langle\psi_1(\omega_1)\psi_2(\omega_2)\psi_3(\omega_3)\rangle\frac{\mathrm{i}\pi[\delta(\omega_2)-\delta(\omega_1)]}{\omega_3}, \tag{4.5}$$

where shorthand notation $\mathrm{d}\omega_{ijl}=\mathrm{d}\omega_i\mathrm{d}\omega_j\mathrm{d}\omega_l\ldots$ is employed and the Fourier representation of the fields over the frequency is used. To calculate the above terms to the fourth order in wave amplitude, we need to account for the nonlinear wave interactions. The calculation is facilitated by observing that the terms in (4.5) are special: they all contain the field amplitude at the zero frequency, $\psi(\boldsymbol{k}, \omega=0)$. Note that the value of the random wave field at the zero frequency also plays an important role in the diffusion of the passive scalar. In that problem if the field vanishes at the zero frequency field, then there is no turbulent diffusion at the order ϵ^2, see Herterich & Hasselmann (1982), Weichman & Glazman (2000) and Balk (2001).

We assume that the force that sustains the stationary wave turbulence vanishes at the zero frequency (note however that the first term on the right-hand side of (4.5) vanishes in the Gaussian approximation independently of this assumption, see Balk *et al.* (2004) and Vucelja *et al.* (2007)). Then the non-zero value of $\psi(\boldsymbol{k}, \omega=0)$ is solely due to the presence of nonlinear wave interactions. As a result, $\psi(\boldsymbol{k}, \omega=0)$ is of at least second order in wave amplitude. Below we derive the corresponding expression for $\psi(\boldsymbol{k}, \omega=0)$ in terms of the higher-order terms. Substituting the resulting expressions into the correlation functions allows the use of Wick's theorem to complete the calculation.

4.3. *The expression for* $\psi(\boldsymbol{k}, \omega=0)$

To calculate λ we need to know $\psi(\boldsymbol{k}, \omega=0)$ to the third order in wave amplitude. Consider the dynamical equation of the surface elevation η, see Batchelor (1967) and Zakharov (1968),

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}[z=\eta] - \nabla\eta\nabla\phi[z=\eta]. \tag{4.6}$$

To order ϵ^2, the equation is

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_0}{\partial z}[z=0] + \eta\frac{\partial^2 \phi_0}{\partial z^2}[z=0] + \frac{\partial \phi_1}{\partial z} - \nabla\eta\nabla\phi_0[z=0] + O(\eta^3), \tag{4.7}$$

where ϕ_0 and ϕ_1 are the terms of the expansion of the potential with respect to the surface elevation, see Appendix A and Zakharov (1968). Using the expressions for ϕ_i, performing the Fourier transform over the space and the time coordinates, and setting the frequency $\omega=0$, we find

$$\begin{aligned} 0 = \overline{|k|}\psi(\boldsymbol{k}, \omega=0) &+ \int \frac{\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\omega}{(2\pi)^3} k_1^2 \psi(\boldsymbol{k}_1, \omega)\eta(\boldsymbol{k}-\boldsymbol{k}_1, -\omega) \\ &- \int \frac{\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\omega}{(2\pi)^3} \overline{|k|}\,\overline{|k_1|}\psi(\boldsymbol{k}_1, \omega)\eta(\boldsymbol{k}-\boldsymbol{k}_1, -\omega) + \int \frac{\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\omega}{(2\pi)^3} \boldsymbol{k}_1 \cdot (\boldsymbol{k}-\boldsymbol{k}_1)\psi(\boldsymbol{k}_1, \omega)\eta(\boldsymbol{k}-\boldsymbol{k}_1, -\omega), \end{aligned}$$

where we have neglected terms of order ϵ^3. It follows that at this order $\psi(\boldsymbol{k}, \omega=0)$ is given by

$$\psi(\boldsymbol{k}, \omega=0) = \int \frac{\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\omega}{(2\pi)^3} \left(\overline{|k_1|} - \frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}}{\overline{|k|}} \right) \psi(\boldsymbol{k}_1, \omega)\eta(\boldsymbol{k}-\boldsymbol{k}_1, -\omega). \tag{4.8}$$

The physical meaning of the above representation is that the zero frequency field arises due to the nonlinear interactions only, and at the lowest order it can be represented as a result of single wave scattering. The above formula suffices to calculate the last two terms in (4.5). Indeed, after we substitute into these terms expression (4.8) for $\psi(\boldsymbol{k}, \omega=0)$, we obtain the correlation function of four fields, which again can be calculated using Wick's theorem. The corresponding calculation is straightforward but cumbersome and it is given in Appendix D. As a result of the calculation these terms vanish identically, leaving

$$\lambda = \frac{1}{2}\int \frac{k_1^2 k_2^2 \mathrm{d}\boldsymbol{k}_{12}\mathrm{d}\omega}{(2\pi)^5}\langle \psi_1(\omega)\psi_2(\omega=0)\rangle. \tag{4.9}$$

To calculate the above quadratic term we again use the fact that it involves $\psi(\boldsymbol{k}, \omega=0)$ by slightly modifying the above computation.

4.4. *Calculation of the quadratic term*

To calculate the right-hand side of (4.9) we note that it is sufficient to know $\psi(\boldsymbol{k}_1, \omega)$ at an arbitrarily small but finite ω where the forcing is again negligible and the dynamic equation (4.6) can be used without the force. The Fourier transform of (4.7) now taken at a small but finite frequency, neglecting terms of order ϵ^3, gives

$$\begin{aligned} \mathrm{i}\omega\eta(\boldsymbol{k},\omega) &= \overline{|k|}\psi(\boldsymbol{k},\omega) + \int \frac{\mathrm{d}\boldsymbol{k}_1\mathrm{d}\omega_1}{(2\pi)^3} k_1^2\psi_1(\omega_1)\eta(\boldsymbol{k}-\boldsymbol{k}_1,\omega-\omega_1) \\ &\quad - \int \frac{\mathrm{d}\boldsymbol{k}_1\mathrm{d}\omega_1}{(2\pi)^3}\overline{|k|}\,\overline{|k_1|}\psi_1(\omega)\eta(\boldsymbol{k}-\boldsymbol{k}_1,\omega-\omega_1) \\ &\quad + \int \frac{\mathrm{d}\boldsymbol{k}_1\mathrm{d}\omega_1}{(2\pi)^3}\boldsymbol{k}_1\cdot(\boldsymbol{k}-\boldsymbol{k}_1)\psi_1(\omega)\eta(\boldsymbol{k}-\boldsymbol{k}_1,\omega-\omega_1), \end{aligned} \tag{4.10}$$

resulting in

$$\psi(\boldsymbol{k},\omega) = \frac{\mathrm{i}\omega\eta(\boldsymbol{k},\omega)}{\overline{|k|}} + \int \frac{\mathrm{d}\boldsymbol{k}_1\mathrm{d}\omega_1}{(2\pi)^3}\left(\overline{|k_1|} - \frac{\boldsymbol{k}_1\cdot\boldsymbol{k}}{\overline{|k|}}\right)\psi(\boldsymbol{k}_1,\omega_1)\eta(\boldsymbol{k}-\boldsymbol{k}_1,\omega-\omega_1) + O(\eta^3).$$

It follows that (4.9) can be written

$$\begin{aligned} \lambda &= \frac{1}{2}\int \frac{k_1^2k_2^2\mathrm{d}\boldsymbol{k}_{12}}{(2\pi)^4}\int\frac{\mathrm{d}\omega}{2\pi}\Bigg\langle\Bigg[\frac{\mathrm{i}\omega\eta(\boldsymbol{k}_1,\omega)}{\overline{|k_1|}} + \int\frac{\mathrm{d}\boldsymbol{k}_3\mathrm{d}\omega_1}{(2\pi)^3}\left(\overline{|k_3|} - \frac{\boldsymbol{k}_3\cdot\boldsymbol{k}_1}{\overline{|k_1|}}\right)\psi(\boldsymbol{k}_3,\omega_1) \\ &\quad \times\eta(\boldsymbol{k}_1-\boldsymbol{k}_3,\omega-\omega_1) + O(\eta^3)\Bigg]\psi(\boldsymbol{k}_2,\omega=0)\Bigg\rangle \\ &= \frac{1}{2}\int \frac{k_1^2k_2^2\mathrm{d}\boldsymbol{k}_{12}}{(2\pi)^4}\int\frac{\mathrm{d}\omega_1}{2\pi}\Bigg\langle\Bigg[\int\frac{\mathrm{d}\boldsymbol{k}_3\mathrm{d}\omega_3}{(2\pi)^3} \\ &\quad \times\left(\overline{|k_3|} - \frac{\boldsymbol{k}_3\cdot\boldsymbol{k}_1}{\overline{|k_1|}}\right)\psi(\boldsymbol{k}_3,\omega_3)\eta(\boldsymbol{k}_1-\boldsymbol{k}_3,\omega_1-\omega_3) + O(\eta^3)\Bigg]\psi(\boldsymbol{k}_2,\omega=0)\Bigg\rangle, \end{aligned} \tag{4.11}$$

where we used $\omega\langle\eta(\boldsymbol{k}_1,\omega)\psi(\boldsymbol{k}_2,\omega=0)\rangle \propto \omega\delta(\omega)=0$. Next, because $\psi(\boldsymbol{k},\omega=0)$ is by itself of order ϵ^2, see the previous subsection, we conclude that the $O(\eta^3)$ term in (4.11) can be neglected. The physical reason for the possibility of such a neglect is that the quadratic term in fact contains correlations of two low-frequency fields where each one is of the order ϵ^2. In the remaining expression, using (4.8), taking the integral over ω_3 and omitting the terms supported at the zero frequency, we find

$$\begin{aligned} \lambda &= \frac{1}{8}\int\frac{\mathrm{d}\boldsymbol{k}_{13}}{(2\pi)^3}(\boldsymbol{k}_1+\boldsymbol{k}_3)^4\left(\overline{|k_3|} - \frac{\boldsymbol{k}_3\cdot(\boldsymbol{k}_1+\boldsymbol{k}_3)}{\overline{|\boldsymbol{k}_1+\boldsymbol{k}_3|}}\right)[n_3n_{-1}\delta(\Omega_{-1}-\Omega_3) \\ &\quad + n_{-3}n_1\delta(\Omega_1-\Omega_{-3})]\left(\overline{|k_3|} - \overline{|k_1|} + \frac{k_1^2-k_3^2}{\overline{|\boldsymbol{k}_1+\boldsymbol{k}_3|}}\right) = 0. \end{aligned} \tag{4.12}$$

The vanishing of the above expression can be easily verified by noting that the δ-functions imply $k_1 = k_3$. Thus we obtain that the sum of Lyapunov exponents for weakly interacting surface waves is identically zero at the fourth order in wave amplitude.

5. Discussion

We have shown that the sum of the Lyapunov exponents for surface wave turbulence vanishes at the fourth order in wave amplitude. The result holds for arbitrary fluid depth. Then, using the approximate Gaussianity of the statistics, it is easy to see that the leading-order term in λ is of the sixth order in wave amplitude, or higher. Therefore we have derived that $\lambda \lesssim \Omega_k \epsilon^6$. For waves with a typical period of the order of seconds and with not too small $\epsilon = 0.1$, we find that the time scale $1/\Omega_k \epsilon^6$ is of the order of weeks. Thus, even if there is no degeneracy at the sixth order and $\lambda \sim \Omega_k \epsilon^6$, the formation of the inhomogeneities would occur at the time scale of weeks and larger. It is unlikely that a low-amplitude wave turbulence would persist for such a time. Thus we expect the turbulence of small-amplitude surface waves to have a negligible effect on the formation of the floater inhomogeneities in realistic situations. Let us stress that this weak compressibility of surface waves holds in the sense of the long-time action of the flow on the particles, while the characteristic value of the ratio of the surface flow divergence to the curl is of order one.

Let us consider some estimates for the Lyapunov exponents of the floater velocity field. For non-interacting Gaussian surface waves, λ_1 is non-zero at the fourth order in wave amplitude, see Balk *et al.* (2004), while λ is non-zero at the sixth order, Balk *et al.* (2004) and M. G. Stepanov (2006, personal communication). The nonlinear wave interactions and the nonlinearity of the velocity–amplitude relation add to λ_1 additional terms of the fourth order in wave amplitude and higher. It is highly implausible that these terms produce exact cancellation of λ_1 at the fourth order– such a cancellation would depend on the precise form of both the interactions and the velocity–amplitude relation. Moreover, a positive Lyapunov exponent and Lagrangian chaos hold even for rather simple Eulerian flows, see e.g. Bohr & Hansen (1996), so no degeneracy for λ_1 is expected that would lead to an exact cancellation at the fourth order. Therefore we expect that λ_1 for surface waves is of the fourth order in wave amplitude. Similarly, we expect no exact cancellation of the non-vanishing Gaussian terms for λ, at the sixth order in wave amplitude, see also below. Then we have the following order-of-magnitude estimates: $\lambda_1 \propto \Omega_k \epsilon^4$ and $\lambda \propto \Omega_k \epsilon^6$. It follows that surface wave turbulence is also weakly compressible in the sense of the ratio $\lambda/\lambda_1 \ll 1$, which is of the second order in wave amplitude. The Kaplan–Yorke dimension of the particle attractor on the surface is close to the space dimension: $D_{KY} \approx 2 - \lambda/\lambda_1 \approx 2$. For dimensionless exponents $\tilde{\lambda}_1 \equiv \lambda_1/\Omega_k$ and $\tilde{\lambda} \equiv \lambda/\Omega_k$ we find the order-of-magnitude relation $\tilde{\lambda} \sim \tilde{\lambda}_1^{3/2}$ holding at small wave amplitudes. The above estimates are supported by numerical simulations performed by Umeki (1992) in a similar problem with *standing* surface waves. It was shown there that both $\lambda \ll \lambda_1$ and $2 - D_{KY} \ll 1$ hold. Moreover, the numerical values of the dimensionless exponents $\tilde{\lambda}$, $\tilde{\lambda}_1$ found there can be easily seen to be in agreement with the relation $\tilde{\lambda} \sim \tilde{\lambda}_1^{3/2}$. The box-counting dimension of the particle attractor on the surface was found to be very close to D_{KY}, which we expect to hold for wave turbulence as well. The detailed calculations of the exact expressions for λ_1 and λ are subjects for future work.

We believe that our conclusion on the weak clustering in surface waves with a small amplitude is an important step in identifying possible reasons for the clusters of floaters observed ubiquitously on liquid surfaces. Our results suggest that other mechanisms need to be explored, such as wave breaking and Langmuir circulation, see e.g. Thorpe (2005), interplay of waves and currents, see Vucelja *et al.* (2007) and wave field inhomogeneities, see Zakharov (1985) and Crawford, Saffman & Yuen (1980).

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Appendix A. The velocity field of floaters

Here we derive the velocity field of the floaters $\boldsymbol{v}(x, y, t)$ up to the third order in wave amplitude, which is sufficient to calculate λ to the fourth order in wave amplitude. We perform the calculation for arbitrary fluid depth h. Equation (3.1) from the main text can be rewritten as

$$\boldsymbol{v}(x, y, t) = \left(\frac{\partial\psi}{\partial x} - \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial z}[z = \eta(x, y, t)], \frac{\partial\psi}{\partial y} - \frac{\partial\eta}{\partial y}\frac{\partial\phi}{\partial z}[z = \eta(x, y, t)]\right). \quad \text{(A 1)}$$

We observe that in order to establish the expression for $\boldsymbol{v}$ in terms of ψ and η up to the third order in wave amplitude, we need to find the expression for the potential ϕ up to the second order in wave amplitude. To do this we note that $\phi(x, y, z, t)$ satisfies Laplace equation $\nabla^2\phi + \partial_z^2\phi = 0$ with the boundary conditions $\phi(x, y, \eta(x, y, t), t) = \psi(x, y, t)$ and $\partial_z\phi(z = -h) = 0$, see e.g. Landau & Lifshitz (2000). Here ∇ is the two-dimensional gradient operator. To the lowest order in wave amplitude the boundary condition $\phi(x, y, \eta(x, y, t), t) = \psi(x, y, t)$ can be substituted by $\phi(x, y, z = 0, t) = \psi(x, y, t)$. This gives the following expression for the lowest-order approximation to ϕ:

$$\phi_0(x, y, z, t) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^2}\frac{\cosh[k(z + h)]}{\cosh[kh]}\exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}]\psi(\boldsymbol{k}, t), \quad \text{(A 2)}$$

where $\psi(\boldsymbol{k}, t)$ is the Fourier transform of $\psi(x, y, t)$ and all the vectors above are two-dimensional, e.g. $\boldsymbol{r} = (x, y)$. To find the next-order correction ϕ_1 we use the identity

$$\phi(x, y, z, t) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^2}\frac{\cosh[k(z + h)]}{\cosh[kh]}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\int \mathrm{d}\boldsymbol{r}'\phi(\boldsymbol{r}', z = 0, t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}, \quad \text{(A 3)}$$

where $\phi(\boldsymbol{r}', z = 0, t)$ is the exact potential at the plane $z = 0$. Using

$$\phi(x, y, z, t) = \phi(x, y, \eta(x, y, t), t) + [z - \eta(x, y, t)]\left.\frac{\partial\phi(x, y, z, t)}{\partial z}\right|_{z=\eta(x,y,t)} + O[(z - \eta)^2],$$

we find

$$\phi(x, y, z = 0, t) = \psi(x, y, t) - \eta(x, y, t)\frac{\partial\phi_0(x, y, z, t)}{\partial z}[z = 0] + O(\eta^2). \quad \text{(A 4)}$$

Substituting the above into (A 3) we obtain that the second-order contribution to the potential is

$$\phi_1 = -\int \frac{\mathrm{d}\boldsymbol{k}_{123}}{(2\pi)^4}\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\frac{\cosh[|k_1|(z + h)]}{\cosh[|k_1|h]}\exp[\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}]|k_2|\tanh(|k_2|h)\psi_2\eta_3,$$

where we use the same shorthand notation as in § 3. In the approximation of infinitely deep fluid, $h \to \infty$, the above expression reproduces the result given in Zakharov (1968) (there is however a sign difference in the expressions – our sign can be verified by checking that the boundary condition $\phi(x, y, \eta(x, y), t) = \psi(x, y, t)$ is satisfied at the considered order). Using the above expression we find that the first-, second- and third-order contributions to the velocity field of the floaters are given respectively by

$$\left.\begin{aligned}
\boldsymbol{v}_0 &= \nabla\psi = \mathrm{i}\int \frac{\mathrm{d}\boldsymbol{k}_1}{(2\pi)^2}\boldsymbol{k}_1 \exp[\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}]\psi_1, \\
\boldsymbol{v}_1 &= -\left.\frac{\partial\phi_0}{\partial z}\right|_{z=0}\nabla\eta = -\mathrm{i}\int \frac{\mathrm{d}\boldsymbol{k}_{12}}{(2\pi)^4}\exp[\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{r}]\overline{|k_1|}\boldsymbol{k}_2\psi_1\eta_2, \\
\boldsymbol{v}_2 &= -\nabla\eta\left(\eta\left.\frac{\partial^2\phi_0}{\partial z^2}\right|_{z=0} + \left.\frac{\partial\phi_1}{\partial z}\right|_{z=0}\right) = -\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}\boldsymbol{k}_{123}}{(2\pi)^6}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\cdot\boldsymbol{r}}\psi_1\eta_2\eta_3 \\
&\quad\times\left(|k_1|^2\boldsymbol{k}_2 + |k_1|^2\boldsymbol{k}_3 - 2\overline{\sqrt{k_1^2+k_2^2}}\,\overline{|k_1|}\boldsymbol{k}_3\right),
\end{aligned}\right\} \qquad (\mathrm{A}\,5)$$

where we have introduced the shorthand notation $\overline{k} \equiv |k|\tanh(|k|h)$. The above expression is equivalent to (3.2) in the main text.

Appendix B. Wick's theorem

Our exposition of Wick's theorem follows Reichl (1988) and Zinn-Justin (2002). For simplicity we consider first the vector case, then generalizing to the case of the fields. Let α_i be a d-dimensional random Gaussian vector. We assume with no loss of generality that the average $\langle\alpha_i\rangle$ vanishes, which can always be achieved by a shift. Then the probability distribution function of $\boldsymbol{\alpha}$ is given by

$$P(\boldsymbol{\alpha}) = C\exp\left[-\tfrac{1}{2}\alpha_i\Gamma_{ij}^{-1}\alpha_j\right], \qquad (\mathrm{B}\,1)$$

where C is a normalization constant and Γ_{ij} is, as will become clear below, the symmetric matrix of correlations. Following Zinn-Justin (2002) we introduce the characteristic function of the imaginary argument, $F(\boldsymbol{b}) \equiv \langle\exp[\boldsymbol{b}\cdot\boldsymbol{\alpha}]\rangle$. Performing the Gaussian integral with $P(\boldsymbol{\alpha})$ given by (B 1), we obtain

$$F(\boldsymbol{b}) = \exp\left[\tfrac{1}{2}b_i\Gamma_{ij}b_j\right], \qquad (\mathrm{B}\,2)$$

see Reichl (1988) and Zinn-Justin (2002). The absence of any normalization constant above follows from the condition that the average of unity is one, $F(0) = 1$. The moments of $\boldsymbol{\alpha}$ can be found from the characteristic function by differentiation:

$$\langle\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_n}\rangle = \frac{\partial}{\partial b_{i_1}}\frac{\partial}{\partial b_{i_2}}\cdots\frac{\partial}{\partial b_{i_n}}\langle\exp[\boldsymbol{b}\cdot\boldsymbol{\alpha}]\rangle|_{\boldsymbol{b}=0} = \left.\frac{\partial^n\exp\left[b_i\Gamma_{ij}b_j/2\right]}{\partial b_{i_1}\partial b_{i_2}\dots\partial b_{i_n}}\right|_{\boldsymbol{b}=0}. \qquad (\mathrm{B}\,3)$$

All the odd moments vanish while for the pair-correlation function we find $\langle\alpha_i\alpha_j\rangle = \Gamma_{ij}$. Using this expression for Γ_{ij}, we find for the fourth-order moment that

$$\langle\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\alpha_{i_4}\rangle = \langle\alpha_{i_1}\alpha_{i_2}\rangle\langle\alpha_{i_3}\alpha_{i_4}\rangle + \langle\alpha_{i_1}\alpha_{i_3}\rangle\langle\alpha_{i_2}\alpha_{i_4}\rangle + \langle\alpha_{i_1}\alpha_{i_4}\rangle\langle\alpha_{i_2}\alpha_{i_3}\rangle. \qquad (\mathrm{B}\,4)$$

Analogous computations with higher-order moments lead to Wick's theorem: for a Gaussian distribution, the average of a product of $2n$ stochastic variables is equal to the sum of all possible combinations of pairwise averages of the stochastic variables, see Reichl (1988). The generalization of Wick's theorem to the case of a random Gaussian field, rather than a vector, is straightforward: the average of a

$2n$-point correlation function is equal to the sum of all possible combinations of pair correlations, see Zinn-Justin (2002). In the text we use Wick's theorem for the case of fourth-order correlation function, employing the counterpart of (B 4).

Appendix C. Calculation of terms in λ involving products of four fields

In this Appendix we calculate those terms in λ_2 and λ_3 that contain products of four fields and allow the direct use of Wick's theorem. We first consider λ_2 in (2.3).

C.1. *Calculation of the fourth-order terms in λ_2*

Using the explicit form of the surface velocity (3.2) we obtain

$$
\begin{aligned}
2\lambda_2 = &\int \frac{k_1^2 k_2^2 \mathrm{d}\boldsymbol{k}_{12}\mathrm{d}t}{(2\pi)^4}\langle\psi_1(0)\psi_2(t)\rangle - 2\int \frac{\mathrm{d}\boldsymbol{k}_{123}\mathrm{d}t}{(2\pi)^6} k_3^2\left(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2\right)\langle\psi_1(0)\eta_2(0)\psi_3(t)\rangle\overline{|k_1|} \\
&- \int \frac{\mathrm{d}\boldsymbol{k}_{1234}\mathrm{d}t}{(2\pi)^8}\langle\psi_1(0)\eta_2(0)\eta_3(0)\psi_4(t)\rangle k_4^2\Bigg(\overline{|k_1|}^2\left(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2 + \boldsymbol{k}_2\cdot\boldsymbol{k}_3\right) \\
&+ \left(\overline{|k_1|}^2 - 2\overline{\sqrt{k_1^2+k_2^2}}\,\overline{|k_1|}\right)\left(\boldsymbol{k}_1\cdot\boldsymbol{k}_3 + \boldsymbol{k}_2\cdot\boldsymbol{k}_3 + k_3^2\right)\Bigg) \\
&+ \int \frac{\mathrm{d}\boldsymbol{k}_{1234}\mathrm{d}t}{(2\pi)^8}\langle\psi_1(0)\eta_2(0)\psi_3(t)\eta_4(t)\rangle\overline{|k_1|}\;\overline{|k_3|}\left(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2\right)\left(\boldsymbol{k}_3\cdot\boldsymbol{k}_4 + k_4^2\right).
\end{aligned}
$$

We note that the third term above, which can be decomposed by Wick's theorem, vanishes because it is supported at the zero frequency ω_4 imposing $k_4 = 0$. The last term can also be analysed using Wick's theorem. Noting that the pair-correlations are supported at the zero sum of the wavenumbers involved, we find that the last term equals

$$
\begin{aligned}
\int \frac{\mathrm{d}\boldsymbol{k}_{1234}\mathrm{d}t}{(2\pi)^8}\,&\left[\langle\psi_1(0)\psi_3(t)\rangle\langle\eta_2(0)\eta_4(t)\rangle + \langle\psi_1(0)\eta_4(t)\rangle\langle\eta_2(0)\psi_3(t)\rangle\right] \\
&\times \overline{|k_1|}\;\overline{|k_3|}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2)(\boldsymbol{k}_3\cdot\boldsymbol{k}_4 + k_4^2).
\end{aligned} \tag{C 1}
$$

Using the correlation functions from (3.5) and noting the vanishing of the terms containing δ-functions supported at the zero frequency, we can write (C 1) as

$$
\begin{aligned}
\int \frac{\mathrm{d}\boldsymbol{k}_{12}}{(2\pi)^3}\Bigg[&\overline{|k_1|}^2\left(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2\right)^2\left(\frac{\Omega_1 k_2}{4k_1\Omega_2}\right)\left[n_1 n_{-2}\delta(\Omega_1 - \Omega_{-2}) + n_{-1}n_2\delta(\Omega_{-1} - \Omega_2)\right] \\
&-\tfrac{1}{4}\overline{|k_1|}\;\overline{|k_2|}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2)\left(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_1^2\right)\left[n_1 n_{-2}\delta(\Omega_1 - \Omega_{-2}) + n_{-1}n_2\delta(\Omega_{-1} - \Omega_2)\right]\Bigg] = 0,
\end{aligned}
$$

where we have introduced the shorthand notation: $n(\pm\boldsymbol{k}_i, t) = n_{\pm i}(t)$ and $\Omega_{\pm\boldsymbol{k}_i} = \Omega_{\pm i}$. Above, we used that for surface waves $\Omega_{\boldsymbol{k}}$ is an increasing function of $|\boldsymbol{k}|$ and that δ-functions imply $\Omega_1 = \Omega_2$ and $k_1^2 = k_2^2$. We find that λ_2 can be written as

$$
\lambda_2 = \int \frac{\mathrm{d}\boldsymbol{k}_{12}\mathrm{d}t}{2(2\pi)^4} k_1^2 k_2^2\langle\psi_1(0)\psi_2(t)\rangle - \int \frac{\mathrm{d}\boldsymbol{k}_{123}\mathrm{d}t}{(2\pi)^6}\overline{|k_1|}k_3^2\left(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2\right)\langle\psi_1(0)\eta_2(0)\psi_3(t)\rangle. \tag{C 2}
$$

The calculation of the above terms requires taking account of the interactions. We now consider the terms in λ_3 that can be calculated by the direct use of Wick's theorem.

C.2. *Calculation of the fourth-order terms in* λ_3

We consider λ_3 from (2.3). Using the expression (3.2) for the velocity, we obtain

$$\begin{aligned}\lambda_3 = &-\int \frac{\mathrm{d}\boldsymbol{k}_{123}}{(2\pi)^6} k_1^2 k_2^2 (\boldsymbol{k}_2\cdot\boldsymbol{k}_3) \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}t_1 \langle \psi_1(0)\psi_2(t)\psi_3(t_1)\rangle \\ &+ \int \frac{\mathrm{d}\boldsymbol{k}_{1234}}{(2\pi)^8} \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}t_1 \Big[\overline{|k_1|}\big(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2\big) k_3^2 \langle \psi_1(0)\eta_2(0)\psi_3(t)\psi_4(t_1)\rangle \\ &\times (\boldsymbol{k}_3\cdot\boldsymbol{k}_4) + k_1^2 \overline{|k_2|}(\boldsymbol{k}_2\cdot\boldsymbol{k}_3 + k_3^2)\big(\boldsymbol{k}_2\cdot\boldsymbol{k}_4 + \boldsymbol{k}_3\cdot\boldsymbol{k}_4\big)\langle \psi_1(0)\psi_2(t)\eta_3(t)\psi_4(t_1)\rangle \\ &+ k_1^2 k_2^2 (\boldsymbol{k}_2\cdot\boldsymbol{k}_4)\overline{|k_3|}\langle \psi_1(0)\psi_2(t)\psi_3(t_1)\eta_4(t_1)\rangle \Big]. \end{aligned} \tag{C 3}$$

We can use Wick's theorem for the last three terms. Employing the identity

$$\int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}t_1 \exp[\mathrm{i}\omega_1 t + \mathrm{i}\omega_2 t_1] = \frac{\mathrm{i}\pi[\delta(\omega_1) - \delta(\omega_1+\omega_2)]}{\omega_2}, \tag{C 4}$$

We find that part of the terms obtained contain δ-functions supported at the zero frequency and they vanish because of the vanishing of the integrand there. Let us consider the rest of the terms. For the first of the fourth-order terms in (C 3) we find

$$\begin{aligned}\int \frac{\mathrm{d}\boldsymbol{k}_{12}}{(2\pi)^4}\overline{|k_1|}\Bigg\{ &|k_1|^2\big(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2\big)\left[\frac{n_{-1}n_2\delta(\Omega_{-1}-\Omega_2)}{\Omega_2} + \frac{n_{-2}n_1\delta(\Omega_{-2}-\Omega_1)}{\Omega_{-2}}\right] \\ &- k_2^2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2)\left[\frac{n_{-1}n_2\delta(\Omega_{-1}-\Omega_2)}{\Omega_{-1}} + \frac{n_{-2}n_1\delta(\Omega_{-2}-\Omega_1)}{\Omega_1}\right]\Bigg\}\frac{\Omega_1\pi}{4k_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) = 0,\end{aligned}$$

where the equality of the frequencies implies the equality of the wavelengths. Analogously, for the second Gaussian term we find

$$\begin{aligned}\int \frac{\Omega_2\overline{|k_2|}\mathrm{d}\boldsymbol{k}_{23}}{(4\pi)^3 k_2}\Bigg\{ &|k_2|^2\big(\boldsymbol{k}_2\cdot\boldsymbol{k}_3 + k_3^2\big)^2\left[\frac{n_{-3}n_2\delta(\Omega_{-3}-\Omega_2)}{\Omega_{-3}} + \frac{n_{-2}n_3\delta(\Omega_{-2}-\Omega_3)}{\Omega_3}\right] \\ &- k_3^2\big(\boldsymbol{k}_2\cdot\boldsymbol{k}_3 + k_3^2\big)\big(\boldsymbol{k}_2\cdot\boldsymbol{k}_3 + k_2^2\big)\left[\frac{n_{-3}n_2\delta(\Omega_{-3}-\Omega_2)}{\Omega_2} + \frac{n_{-2}n_3\delta(\Omega_{-2}-\Omega_3)}{\Omega_{-2}}\right]\Bigg\} = 0.\end{aligned}$$

Finally, the third Gaussian term contains only δ-functions supported at the zero frequencies, so it also produces zero. We conclude that λ_3 can be written as

$$\lambda_3 = -\int \frac{\mathrm{d}\boldsymbol{k}_{123}}{(2\pi)^6} k_1^2 k_2^2 (\boldsymbol{k}_2\cdot\boldsymbol{k}_3) \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}t_1 \langle \psi_1(0)\psi_2(t)\psi_3(t_1)\rangle. \tag{C 5}$$

C.3. *Summary*

As a result of the calculation in the previous subsections, adding equations (C 2) and (C 5), one can write λ as a sum over the terms the calculation of which involves the wave interactions. Using Fourier representation over the frequency we find

$$\begin{aligned}\lambda = &\frac{1}{2}\int \frac{k_1^2 k_2^2 \mathrm{d}\boldsymbol{k}_{12}\mathrm{d}\omega}{(2\pi)^5}\langle \psi_1(\omega)\psi_2(\omega=0)\rangle \\ &- \int \frac{\mathrm{d}\boldsymbol{k}_{123}\mathrm{d}\omega_{12}}{(2\pi)^8}\overline{|k_1|}k_3^2\big(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_2^2\big)\langle \psi_1(\omega_1)\eta_2(\omega_2)\psi_3(\omega=0)\rangle \\ &- \int \frac{\mathrm{d}\boldsymbol{k}_{123}\mathrm{d}\omega_{123}}{(2\pi)^9} k_1^2 k_2^2 (\boldsymbol{k}_2\cdot\boldsymbol{k}_3)\langle \psi_1(\omega_1)\psi_2(\omega_2)\psi_3(\omega_3)\rangle \frac{\mathrm{i}\pi[\delta(\omega_2)-\delta(\omega_1)]}{\omega_3},\end{aligned} \tag{C 6}$$

where we have introduced shorthand notation $d\omega_{ijl} = d\omega_i d\omega_j d\omega_l \ldots$ and used in the last term the identity (C 4) and the proportionality of the correlation function to $\delta(\omega_1 + \omega_2 + \omega_3)$.

Appendix D. Interaction terms containing the products of three fields

To calculate the interaction terms of the third order we use (4.8), Wick's decomposition and a Fourier transformed version of (3.5). The second term in (C 6) can be written with the help of (4.8) as

$$
\begin{aligned}
&-\int \frac{d\boldsymbol{k}_{1234} d\omega_{123}}{(2\pi)^{11}} \overline{|k_1|} k_3^2 \left(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + k_2^2\right) \left(\overline{|k_4|} - \frac{\boldsymbol{k}_4 \cdot \boldsymbol{k}_3}{\overline{|k_3|}}\right) \langle \psi(\boldsymbol{k}_1, \omega_1) \eta(\boldsymbol{k}_2, \omega_2) \psi(\boldsymbol{k}_4, \omega_3) \\
&\times \eta(\boldsymbol{k}_3 - \boldsymbol{k}_4, -\omega_3)\rangle = -\int \frac{d\boldsymbol{k}_{12} d\omega_1}{(2\pi)^3} \overline{|k_1|} (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2 \left(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + k_2^2\right) \left(\overline{|k_1|} - \frac{\boldsymbol{k}_1 \cdot (\boldsymbol{k}_1 + \boldsymbol{k}_2)}{\overline{|\boldsymbol{k}_1 + \boldsymbol{k}_2|}}\right) \\
&\times \left(\frac{\Omega_1 |k_2|}{4\Omega_2 |k_1|}\right) [n_1 \delta(\omega_1 + \Omega_1) + n_{-1} \delta(\omega_1 - \Omega_{-1})][n_2 \delta(-\omega_1 + \Omega_2) + n_{-2} \delta(\omega_1 + \Omega_{-2})] \\
&+ \frac{1}{4} \int \frac{d\boldsymbol{k}_{12} d\omega_1}{(2\pi)^3} \overline{|k_1|} (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2 \left(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + k_2^2\right) \left(\overline{|k_2|} - \frac{\boldsymbol{k}_2 \cdot (\boldsymbol{k}_1 + \boldsymbol{k}_2)}{\overline{|\boldsymbol{k}_1 + \boldsymbol{k}_2|}}\right) \\
&\times [n_1 \delta(\omega_1 + \Omega_1) - n_{-1} \delta(\omega_1 - \Omega_{-1})][n_{-2} \delta(\omega_1 + \Omega_{-2}) - n_2 \delta(\omega_1 - \Omega_2)],
\end{aligned} \tag{D 1}
$$

where to show that the remaining contraction vanishes, we can use that $\langle \psi(\boldsymbol{k}, \omega = 0)\rangle$ is representable as an integral over $\langle \psi(\boldsymbol{k}, t) \eta(\boldsymbol{k}', t)\rangle$, which in the Gaussian approximation vanishes by (3.5). Using $\Omega_{-\boldsymbol{k}} = \Omega_{\boldsymbol{k}}$ and noting that the terms supported at $\Omega_{\boldsymbol{k}} = 0$ vanish, we can rewrite as

$$
\begin{aligned}
&-\frac{1}{4} \int \frac{d\boldsymbol{k}_{12}}{(2\pi)^3} \overline{|k_1|} (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2 (\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + k_2^2) \left(\overline{|k_1|} - \frac{\boldsymbol{k}_1 \cdot (\boldsymbol{k}_1 + \boldsymbol{k}_2)}{\overline{|\boldsymbol{k}_1 + \boldsymbol{k}_2|}}\right) [n_1 n_{-2} \delta(\Omega_1 - \Omega_{-2}) \\
&+ n_{-1} n_2 \delta(\Omega_2 - \Omega_{-1})] + \frac{1}{4} \int \frac{d\boldsymbol{k}_{12}}{(2\pi)^3} \overline{|k_1|} (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2 \left(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + k_2^2\right) \left(\overline{|k_2|} - \frac{\boldsymbol{k}_2 \cdot (\boldsymbol{k}_1 + \boldsymbol{k}_2)}{\overline{|\boldsymbol{k}_1 + \boldsymbol{k}_2|}}\right) \\
&\times [n_1 n_{-2} \delta(\Omega_1 - \Omega_{-2}) + n_{-1} n_2 \delta(\Omega_2 - \Omega_{-1})] = \frac{1}{4} \int \frac{d\boldsymbol{k}_{12}}{(2\pi)^3} \overline{|k_1|} (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2 \left(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + k_2^2\right) \\
&\times \left(\overline{|k_2|} - \overline{|k_1|} + \frac{k_1^2 - k_2^2}{\overline{|\boldsymbol{k}_1 + \boldsymbol{k}_2|}}\right) [n_1 n_{-2} \delta(\Omega_1 - \Omega_{-2}) + n_{-1} n_2 \delta(\Omega_2 - \Omega_{-1})] = 0,
\end{aligned} \tag{D 2}
$$

where we have used that δ-functions imply $\Omega_1 = \Omega_2$ and $k_1 = k_2$. Let us now consider the last term in (C 6) that can be written as

$$
i\pi \int \frac{d\boldsymbol{k}_{123} d\omega_{23}}{\omega_3 (2\pi)^9} k_1^2 k_2^2 \boldsymbol{k}_3 \cdot (\boldsymbol{k}_2 - \boldsymbol{k}_1) \langle \psi(\boldsymbol{k}_1, \omega = 0) \psi(\boldsymbol{k}_2, \omega_2) \psi(\boldsymbol{k}_3, \omega_3)\rangle. \tag{D 3}
$$

Substituting (4.8) we find

$$
\begin{aligned}
&i\pi \int \frac{d\boldsymbol{k}_{1234} d\omega_{234} k_1^2 k_2^2 \boldsymbol{k}_3 \cdot (\boldsymbol{k}_2 - \boldsymbol{k}_1)}{\omega_3 (2\pi)^{12}} \left(\overline{|k_4|} - \frac{\boldsymbol{k}_4 \cdot \boldsymbol{k}_1}{\overline{|k_1|}}\right) \langle \eta(\boldsymbol{k}_1 - \boldsymbol{k}_4, -\omega_4) \psi(\boldsymbol{k}_2, \omega_2) \psi(\boldsymbol{k}_3, \omega_3) \\
&\times \psi(\boldsymbol{k}_4, \omega_4)\rangle = i\pi \int \frac{d\boldsymbol{k}_{23} d\omega_3}{\omega_3 (2\pi)^4} (\boldsymbol{k}_2 + \boldsymbol{k}_3)^2 k_2^2 \boldsymbol{k}_3 \cdot (2\boldsymbol{k}_2 + \boldsymbol{k}_3) \left(\overline{|k_3|} - \frac{\boldsymbol{k}_3 \cdot (\boldsymbol{k}_2 + \boldsymbol{k}_3)}{\overline{|\boldsymbol{k}_2 + \boldsymbol{k}_3|}}\right) \frac{\Omega_3}{4i|k_3|}
\end{aligned}
$$

$$\times\,[n_2\delta(\omega_3-\Omega_2)-n_{-2}\delta(\omega_3+\Omega_{-2})]\,[n_3\delta(\omega_3+\Omega_3)+n_{-3}\delta(\omega_3-\Omega_{-3})]+\mathrm{i}\pi\int\frac{\mathrm{d}\boldsymbol{k}_{23}\mathrm{d}\omega_3}{\omega_3(2\pi)^4}$$
$$\times\,(\boldsymbol{k}_2+\boldsymbol{k}_3)^2k_2^2\boldsymbol{k}_3\cdot(2\boldsymbol{k}_2+\boldsymbol{k}_3)\left(\frac{}{|k_2|}-\frac{\boldsymbol{k}_2\cdot(\boldsymbol{k}_2+\boldsymbol{k}_3)}{|\boldsymbol{k}_2+\boldsymbol{k}_3|}\right)\left(\frac{\Omega_2}{4\mathrm{i}|k_2|}\right)$$
$$\times\,[n_2\delta(\omega_3-\Omega_2)+n_{-2}\delta(\omega_3+\Omega_{-2})]\,[n_3\delta(\omega_3+\Omega_3)-n_{-3}\delta(\omega_3-\Omega_{-3})]\,, \qquad \text{(D 4)}$$

where the term involving the contraction $\langle\eta(\boldsymbol{k}_1-\boldsymbol{k}_4,-\omega_4)\psi(\boldsymbol{k}_4,\omega_4)\rangle$ corresponding to $\langle\psi(\boldsymbol{k},\omega=0)\rangle$ vanishes as was shown in the analysis of the previous term, where we omitted the terms supported at the zero frequency. Taking the integral over ω_3, omitting the terms supported at the zero frequency, we find

$$\mathrm{i}\pi\int\frac{\mathrm{d}\boldsymbol{k}_{23}}{(2\pi)^4}(\boldsymbol{k}_2+\boldsymbol{k}_3)^2k_2^2\boldsymbol{k}_3\cdot(2\boldsymbol{k}_2+\boldsymbol{k}_3)\left(\frac{}{|k_3|}-\frac{\boldsymbol{k}_3\cdot(\boldsymbol{k}_2+\boldsymbol{k}_3)}{|\boldsymbol{k}_2+\boldsymbol{k}_3|}\right)$$
$$\times\left[\frac{n_2n_{-3}\delta(\Omega_2-\Omega_{-3})}{4\mathrm{i}|k_3|}+\frac{n_{-2}n_3\delta(\Omega_{-2}-\Omega_3)}{4\mathrm{i}|k_3|}\right]+\mathrm{i}\pi\int\frac{\mathrm{d}\boldsymbol{k}_{23}}{(2\pi)^4}$$
$$\times(\boldsymbol{k}_2+\boldsymbol{k}_3)^2k_2^2\boldsymbol{k}_3\cdot(2\boldsymbol{k}_2+\boldsymbol{k}_3)\left(\frac{}{|k_2|}-\frac{\boldsymbol{k}_2\cdot(\boldsymbol{k}_2+\boldsymbol{k}_3)}{|\boldsymbol{k}_2+\boldsymbol{k}_3|}\right)$$
$$\times\left[-\frac{n_{-2}n_3\delta(\Omega_3-\Omega_{-2})}{4\mathrm{i}|k_2|}-\frac{n_2n_{-3}\delta(\Omega_2-\Omega_{-3})}{4\mathrm{i}|k_2|}\right]=0, \qquad \text{(D 5)}$$

where zero is obtained in the same way as with the previous term. The result of this Appendix is that λ can be written as

$$\lambda=\frac{1}{2}\int\frac{k_1^2k_2^2\mathrm{d}\boldsymbol{k}_{12}}{(2\pi)^4}\int\frac{\mathrm{d}\omega}{2\pi}\langle\psi(\boldsymbol{k}_1,\omega)\psi(\boldsymbol{k}_2,\omega=0)\rangle. \qquad \text{(D 6)}$$

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